

Problem Set

This problem set is voluntary. Questions 1 and 2 can be solved with pen and paper (plus a few lines of code for one numerical part). Question 3 has an analytical part and an optional computational part.

Non-Cooperative Household Decisions

Question 1: Voluntary Contributions to Household Public Goods

This question is based on Doepke and Tertilt (2019). A wife (f) and a husband (m) make decisions non-cooperatively: each spouse chooses his or her own contributions to household public goods, taking the contributions of the other spouse as given. The solution concept is Nash equilibrium.

Part A: A single public good. Each spouse $g \in \{f, m\}$ has utility

$$\log(c_g) + \log(C),$$

where c_g is private consumption and $C = C_f + C_m$ is a household public good, with $C_g \geq 0$ the contribution of spouse g . Spouse g has income $y_g > 0$ and budget constraint $c_g + C_g = y_g$. Let $Y = y_f + y_m$.

- (a) Derive the best-response function $C_g(C_{-g})$ of each spouse.
- (b) Characterize the Nash equilibrium in which both spouses make positive contributions. Show that c_f , c_m , and C depend only on total income Y , not on its distribution between the spouses. For which distributions of income does such an interior equilibrium exist?
- (c) Now suppose that y_f is below the threshold derived in (b). Characterize the equilibrium. What is the effect of a small transfer from the husband to the wife on C and on the utility of each spouse? Explain why income pooling breaks down.

Part B: Home production and comparative advantage. Now there is a continuum of public goods $i \in [0, 1]$, and utility is

$$\log(c_g) + \int_0^1 \log(C_i) di.$$

Each public good can be produced by either spouse, using goods $E_{g,i}$ and time $T_{g,i}$:

$$C_i = C_{f,i} + C_{m,i}, \quad C_{g,i} = E_{g,i}^{1-\alpha(i)} T_{g,i}^{\alpha(i)},$$

where $\alpha(i) \in [0, 1]$ is non-decreasing in i , so that goods are ordered from the most goods-intensive to the most time-intensive. The goods and time inputs for a given contribution have to come from the same spouse. Spouse g has market wage w_g , non-labor income (wealth) x_g , and a time endowment of one, so that the constraints are

$$c_g + \int_0^1 E_{g,i} di = w_g(1 - T_g) + x_g, \quad \int_0^1 T_{g,i} di = T_g.$$

(Ignore the constraint $T_g \leq 1$.)

- (d) Show that the minimum cost for spouse g of producing one unit of public good i , where time is valued at the market wage, is

$$p_g(i) = \frac{w_g^{\alpha(i)}}{\alpha(i)^{\alpha(i)} (1 - \alpha(i))^{1-\alpha(i)}}.$$

Rewrite the problem of spouse g in terms of full income $w_g + x_g$ and these unit costs.

- (e) Taking the contributions of the other spouse as given, show that if spouse g contributes to good i , then $C_i = c_g/p_g(i)$, so that spouse g spends exactly c_g on each good that he or she provides alone. In equilibrium, which spouse provides a given good i ?
- (f) Suppose first that the spouses have the same wage, $w_f = w_m = w$, so that they have the same unit cost for every public good. Show that in an equilibrium in which both spouses provide some public goods, private consumption and the provision of each public good are not affected by small redistributions of wealth between the spouses. Relate this result to Part A.
- (g) Now let $\alpha(i) = i$ and $w_f < w_m$. Show that there is a cutoff $\bar{i}$ such that the husband provides all goods $i < \bar{i}$ and the wife provides all goods $i > \bar{i}$ ("separate spheres"). Express c_f and c_m as functions of $\bar{i}$, and show that an interior cutoff solves

$$\bar{i} \log \left(\frac{w_m}{w_f} \right) = \log \left(\frac{w_m + x_m}{w_f + x_f} \right) + \log \left(\frac{2 - \bar{i}}{1 + \bar{i}} \right).$$

Show that this equation has a unique solution, and give a condition on wealth under which the cutoff is interior.

- (h) Consider a mandated transfer $\epsilon > 0$ from the husband to the wife, i.e., x_f rises by ϵ and x_m falls by ϵ , and assume that the cutoff is interior before and after the transfer. Show that $\bar{i}$ falls and c_f/c_m rises. What happens to the provision of goods that are always provided by the wife, relative to goods that are always provided by the husband? Explain why the neutrality results of Part A and part (f) fail here.
- (i) Let $w_f = 0.5$, $w_m = 1$, and $x_f = x_m = 0$. Compute $\bar{i}$, c_f , and c_m , and plot the equilibrium provision C_i as a function of i . Repeat for a transfer of $\epsilon = 0.3$ from the husband to the wife. Does the transfer increase the utility derived from public goods, $\int_0^1 \log(C_i) di$?
- (j) Suppose that before the public goods are provided, the husband can make a voluntary lump-sum transfer $X \geq 0$ to the wife. Argue that if the husband chooses $X > 0$, a mandated transfer $\epsilon \leq X$ from husband to wife has no effect on the equilibrium allocation. What does this imply for interpreting the empirical evidence that transfers targeted to women change household spending patterns?

Bargaining without Commitment

Question 2: A Location Decision Subject to Veto

This question is a stylized version of the timing in Doepke and Kindermann (2019). A couple lives for two periods.

- In period 1, the couple chooses a location $\ell \in \{S, N\}$. S is the status quo; moving to N requires the consent of both spouses, i.e., each spouse can veto the move.
- In period 2, spouse g earns wage w_g^ℓ in location ℓ , and the spouses bargain over the division of household consumption. If they cooperate, household consumption is $(1 + \alpha)(w_f^\ell + w_m^\ell)$, where $\alpha > 0$ captures the gains from cooperation (e.g., economies of scale). The threat point is non-cooperation, in which each spouse consumes his or her own earnings, $c_g = w_g^\ell$. The outcome is given by the Nash bargaining solution with equal weights.

Utility is linear in consumption, $u(c_g) = c_g$, and there is no utility from period-1 consumption that depends on the location decision. Wages are:

$$\text{Location } S : w_f^S = w_m^S = 1, \quad \text{Location } N : w_f^N = 1 - \delta, \quad w_m^N = 1 + \Delta,$$

with $\delta, \Delta > 0$. That is, a move raises the husband's wage and lowers the wife's wage.

- (a) Solve for the period-2 bargaining outcome in each location. Show that $c_g^\ell = w_g^\ell + \frac{\alpha}{2} (w_f^\ell + w_m^\ell)$.
- (b) Under which condition does moving to N maximize total household consumption? Show that in this case, moving is Pareto efficient if transfers between the spouses are possible.
- (c) Derive the condition under which the wife consents to the move. Show that for

$$\delta < \Delta < \frac{2+\alpha}{\alpha} \delta,$$

the wife vetoes an efficient move. Show that the husband never vetoes an efficient move. Interpret the result in terms of bargaining power. How does the range of inefficient vetoes depend on α ?

- (d) Suppose that in period 1 the spouses can sign an enforceable contract that specifies a period-2 transfer from the husband to the wife, contingent on moving. Show that the couple moves if and only if the move is efficient. What is the friction that is responsible for the inefficiency in (c)?
- (e) Suppose instead that the threat point in period 2 is divorce, and that after divorce each spouse earns a wage of 1 regardless of the location the couple chose (e.g., because each returns to the hometown). Show that the location decision is now efficient. Explain.
- (f) Return to the original threat point and assume that $\delta < \Delta < \frac{2+\alpha}{\alpha} \delta$. The government offers a relocation subsidy b , paid in period 2 if the couple lives in N . The subsidy is added to the income of the recipient (and hence enters the recipient's threat point). Find the minimum subsidy that induces the move if (i) the subsidy is paid to the wife, and (ii) it is paid to the husband. Explain the difference. Relate your answer to the argument in Doepke and Kindermann (2019) that family policies are more effective if they target the spouse whose consent is pivotal.
- (g) In Doepke and Kindermann (2019), the period-1 decision is whether to have a child, and the cost of children falls disproportionately on the mother. Map this setting into the framework above. Why does the distribution of the cost of children between the parents matter for fertility, beyond the total cost?

Limited Commitment

Question 3: Insurance within Marriage under Limited Commitment

This question develops the limited-commitment model of the household (Kocherlakota 1996; Ligon, Thomas, and Worrall 2002; Marcet and Marimon 2019) that underlies Voena (2015) and Mazzocco (2007), in the simplest setting where it can be solved (almost) analytically.

A married couple lives forever and discounts the future at rate $\beta \in (0, 1)$. There is no saving. Total household income is constant at $Y = y_H + y_L$ with $y_H > y_L > 0$, but its distribution fluctuates. In each period, the state $s_t \in \{1, 2\}$ is drawn i.i.d. with probability $1/2$ each. In state 1, the wife earns y_H and the husband earns y_L ; in state 2, it is the other way around. While married, spouse $j \in \{f, m\}$ has period utility $\log(c_{jt}) + \theta$, where $\theta \geq 0$ is the per-period gain from marriage, and consumption satisfies $c_{ft} + c_{mt} = Y$. Each spouse can unilaterally divorce at the beginning of any period, after observing s_t . Divorce is permanent, there is no remarriage, and a divorced spouse consumes his or her own income in every period.

- (a) Compute the value of divorce $\bar{V}(y)$ for a spouse whose current income is $y \in \{y_H, y_L\}$.
- (b) *Full commitment.* Suppose that at marriage, before s_0 is realized, the spouses commit to a state-contingent plan that maximizes

$$\mu_0 \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [\log(c_{ft}) + \theta] + (1 - \mu_0) \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [\log(c_{mt}) + \theta].$$

Show that $c_{ft}/c_{mt} = \mu_0/(1 - \mu_0)$ in every period. For $\mu_0 = 1/2$, derive a condition on $(\beta, \theta, y_H, y_L)$ under which this full-insurance allocation satisfies both spouses' participation constraints in every period and state, so that it is also attainable without commitment.

- (c) *Limited commitment: the Marcet–Marimon approach.* Now add the participation constraints

$$\mathbb{E}_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} [\log(c_{j\tau}) + \theta] \geq \bar{V}(y_{jt}) \quad \text{for all } t, s^t, j \in \{f, m\}.$$

Set up the Lagrangian with multipliers $\beta^t \pi(s^t) \lambda_j(s^t) \geq 0$ on the participation constraints. Show that the first-order conditions imply

$$\frac{c_{ft}}{c_{mt}} = \frac{M_{ft}}{M_{mt}}, \quad M_{ft} = \mu_0 + \sum_{\tau \leq t} \lambda_{f\tau}, \quad M_{mt} = 1 - \mu_0 + \sum_{\tau \leq t} \lambda_{m\tau}.$$

Explain why the relative weight $\mu_t = M_{ft}/(M_{ft} + M_{mt})$ changes only when a participation constraint binds, and why (μ_{t-1}, s_t) is a sufficient state variable. Write down the recursive formulation in terms of value functions $V_j(\mu, s)$ and the updating rule $\mu_t = g(\mu_{t-1}, s_t)$.

- (d) *Limited commitment: promised utility.* Alternatively, let v be the utility promised to the wife at the beginning of a period (after s is observed), and let $P(v, s)$ be the highest utility the husband can obtain while delivering v to the wife. Write the Bellman equation for $P(v, s)$, where the choice variables are current consumption and continuation promises $v'(s')$, subject to promise keeping and both participation constraints. Derive the first-order and envelope conditions, and show that $-P_v(v, s)$ equals c_f/c_m . Relate this to the weight μ in part (c).
- (e) Suppose that the condition in (b) fails. Conjecture that in the long run, the spouse with high current income consumes $\bar{c}$ and the other spouse consumes $Y - \bar{c}$, with $Y/2 < \bar{c} \leq y_H$. Show that $\bar{c}$ solves

$$\log(\bar{c}) + \theta + \frac{\beta}{1-\beta} \left[\frac{\log(\bar{c}) + \log(Y - \bar{c})}{2} + \theta \right] = \bar{V}(y_H),$$

and argue that the constrained-efficient allocation corresponds to the smallest root in $(Y/2, y_H]$. Starting from $\mu_0 = 1/2$, describe the path of μ_t and of consumption. In what sense does the allocation depend on history, and in what sense is it “memoryless”?

- (f) Let $y_H = 1.5$, $y_L = 0.5$, and $\theta = 0$. Compute $\bar{c}$ for $\beta \in \{0.6, 0.65, 0.7, 0.75\}$, and find the threshold for β above which full insurance is attainable. How do the results change if $\theta = 0.02$? Interpret.
- (g) *Divorce law.* Suppose that divorce law mandates income sharing after divorce, so that a divorced spouse with income y consumes $(1 - \tau)y + \tau Y/2$ every period, with $\tau \in [0, 1]$. For $\beta = 0.7$ and $\theta = 0$, compute $\bar{c}$ for $\tau \in \{0, 0.25, 0.5, 0.75, 1\}$. Explain why more income sharing after divorce can *reduce* insurance within marriage. Relate your findings to the effects of divorce laws and the division of property in Voena (2015).
- (h) *Optional computational part.* Write a program that solves the model for general (Markov) income processes by iterating on the value functions $V_j(\mu, s)$ on a grid for μ : given V_j , compute the bounds $\underline{\mu}(s)$ and $\bar{\mu}(s)$ at which the participation constraints bind, update the policy $g(\mu, s)$, update V_j , and iterate to convergence. Verify your program against your answers in (e) and (f). Then describe (without implementing) how the state space and the algorithm change if the couple can save in a riskless asset and assets are divided equally at divorce, as in Voena (2015).